THE SHUFFLE ALGEBRA REVISITED

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Abstract. In this paper we introduce certain new features of the shuffle algebra of [1] that will allow us to obtain explicit formulas for the isomorphism between its Drinfeld double and the elliptic Hall algebra of [3]. These results are necessary for our work in [2].

1. Introduction

In [3], the authors have constructed an isomorphism Υ between the positive half of the elliptic Hall algebra $\mathcal{E}^+$ and the shuffle algebra $\mathcal{A}^+$. This isomorphism is given by generators and relations, and it extends to the Drinfeld doubles of the algebras in question. Our goal in this paper is to make this isomorphism Υ more explicit, by proving:

Theorem 1.1. *We have:*

$$\Upsilon(u_{k,d}) = P_{k,d} \tag{1.1}$$

where the $u_{k,d}$ are the standard generators of $\mathcal{E}^+$ (see Subsection 6.1 for the definition) and the $P_{k,d}$ are the minimal shuffle elements of Subsection 4.9.

Relation (4.12) gives an explicit formula for $P_{k,d}$, albeit a non-closed one. It will feature in [2], where we will use it to identify $P_{k,d}$ with certain geometric operators that act on the $K-$theory of the moduli spaces of sheaves. Let us say a few things about the structure of this paper:

- In Section 2, we define the shuffle algebra $\mathcal{A}^+$ and introduce the crucial notion of degree.
- In Section 3, we take a notion of limit from [1] and generalize it. It will provide us with certain coproducts, in an appropriate sense, and we will show that they are coassociative and respect the multiplication.
- In Section 4, we introduce the minimality property that defines the shuffle elements $P_{k,d}$. We also introduce the shuffle elements $Q_{k,d}$, which will be important for the proof of Theorem 1.1.
- In Section 5, we present explicit formulas for the $P_{k,d}$ and $Q_{k,d}$.

- In Section 6, we define the elliptic hall algebra $\mathcal{E}^+$ and prove Theorem 1.1.
- In Section 7, we introduce the doubles $\mathcal{A}$ and $\mathcal{E}$ of the above algebras, and prove an analogue of Theorem 1.1 in Theorem 7.7.

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2. The Shuffle Algebra

2.1. We will work over the field $\mathbb{K} = \mathbb{C}(q_1, q_2)$, and let us write for convenience $q = q_1 q_2$. Consider an infinite set of variables $z_1, z_2, ...$, and let us look at the $\mathbb{K}-$vector space:

$$\bigoplus_{k \geq 0} \mathrm{Sym}_{\mathbb{K}}(z_1, ..., z_k), \tag{2.1}$$

bigraded by k and homogenous degree. We can endow it with a $\mathbb{K}-$algebra structure via the **shuffle product**:

$$P(z_1, ..., z_k) * Q(z_1, ..., z_l) =$$

$$= \mathrm{Sym}\left[P(z_1, ..., z_k) Q(z_{k+1}, ..., z_{k+l}) \prod_{i=1}^{k} \prod_{j=k+1}^{k+l} \omega\left(\frac{z_i}{z_j}\right)\right] \tag{2.2}$$

where:

$$\omega(x) = \frac{(x-1)(x-q)}{(x-q_1)(x-q_2)} \tag{2.3}$$

and Sym denotes the symmetrization operator on rational functions. It is normalized such that relation (2.2) can be rewritten as:

$$P(z_1, ..., z_k) * Q(z_1, ..., z_l) = \sum_{\substack{\{1,...,k+l\} = A \sqcup B \\ |A| = k, |B| = l}} P(z_A) Q(z_B) \prod_{a \in A} \prod_{b \in B} \omega\left(\frac{z_a}{z_b}\right) \tag{2.4}$$

where for $A = \{a_1, ..., a_k\}$, we write $P(z_A) = P(z_{a_1}, ..., z_{a_k})$.

2.2. The **shuffle algebra** $\mathcal{A}^+$ (see [1]) is defined as the subspace of (2.1) consisting of rational functions of the form:

$$P(z_1, ..., z_k) = \frac{\prod_{1 \leq i < j \leq k} (z_i - z_j)^2 \cdot p(z_1, ..., z_k)}{\prod_{1 \leq i \neq j \leq k} (z_i - q_1 z_j)(z_i - q_2 z_j)}, \qquad k \geq 1 \tag{2.5}$$

where p is a symmetric Laurent polynomial that satisfies the **wheel conditions**:

$$p(z_1, q_1 z_1, q z_1, z_4, ..., z_k) = p(z_1, q_2 z_1, q z_1, z_4, ..., z_k) = 0 \tag{2.6}$$

This condition is vacuous for $k \leq 2$. We will call elements of $\mathcal{A}^+$ **shuffle elements**.

2.3. The fact that $\mathcal{A}^+$ is an algebra follows from:

Proposition 2.4. *If $P, Q \in \mathcal{A}^+$, then $P * Q \in \mathcal{A}^+$.*

Proof By the definition of multiplication (2.2), for P and Q as in (2.5) we have:

$$P * Q = \frac{\prod_{1 \leq i < j \leq k+l} (z_i - z_j)^2}{\prod_{i,j=1}^{k+l} (z_i - q_1 z_j)(z_i - q_2 z_j)}$$

$$\cdot \operatorname{Sym} \left[p(z_1, ..., z_k) q(z_{k+1}, ..., z_{k+l}) \cdot \prod_{i \leq k < j} \frac{(z_i - q z_j)(z_j - q_1 z_i)(z_j - q_2 z_i)}{z_i - z_j} \right] \quad (2.7)$$

The expression on the last line is a rational function with at most simple poles at $z_j = z_i$. Because it is symmetric, it must necessarily be regular at $z_j = z_i$, and therefore it is a Laurent polynomial in the z variables. To prove that $P * Q \in \mathcal{A}^+$, we need only prove that this expression satisfies the wheel conditions. In fact, we will show that every summand of the Sym in (2.7) satisfies them.

To see this, note that we need to set three of the variables equal to $z, q_1 z, qz$ (or $z, q_2 z, qz$, but this case is analogous) and show that the given summand vanishes. If all three of the variables are among $\{z_1, ..., z_k\}$ or $\{z_{k+1}, ..., z_{k+l}\}$, then the summand vanishes because p and q satisfy the wheel conditions themselves. If one of the variables is in $\{z_1, ..., z_k\}$ and two are in $\{z_{k+1}, ..., z_{k+l}\}$ (or vice-versa) then the product in (2.7) vanishes, and therefore so does the summand.

□

2.5. In [1] and [3], the shuffle algebra is defined as the subalgebra of (2.1) generated by $\operatorname{Sym}_{\mathbb{K}}(z_1)$. We denote this algebra by $\tilde{\mathcal{A}}^+$, and Proposition 2.4 implies that $\tilde{\mathcal{A}}^+ \subset \mathcal{A}^+$. In fact, we believe that the two are equal:

Conjecture 2.6.

$$\tilde{\mathcal{A}}^+ = \mathcal{A}^+$$

We will not prove this conjecture, since it is not essential for the results in this paper. But if we do not accept it, we need to replace the shuffle algebra $\mathcal{A}^+$ by the (a priori smaller) algebra $\tilde{\mathcal{A}}^+$ in all our definitions and statements.

2.7. The shuffle algebra is bigraded by the number of variables k and the total homogenous degree d of shuffle elements:

$$\mathcal{A}^+ = \bigoplus_{k \geq 0} \mathcal{A}_k^+, \qquad \mathcal{A}_k^+ = \bigoplus_{d \in \mathbb{Z}} \mathcal{A}_{k,d}^+$$

Let us consider the linear map $\varphi : \mathcal{A}_{k,d}^+ \longrightarrow \mathbb{K}$ given by:

$$\varphi(P)=\left[P(z_1,...,z_k)\cdot\prod_{1\le i\neq j\le k}\frac{z_i-q_1z_j}{z_i-z_j}\right]_{z_i=q^{-i}} q_1^{\frac{-k^2+kd+d+k}{2}}(q_1-1)^k\prod_{i=1}^k\frac{q_1^{i-1}-q_2}{q_1^i-1}$$

[1] This map will allow us to normalize our shuffle elements, which will be crucial in the proof of Theorem 1.1. We will now prove that it behaves nicely with respect to the shuffle product:

Proposition 2.8. *Given $P\in\mathcal{A}^+_{k,d}$ and $Q\in\mathcal{A}^+_{l,e}$, we have:*

$$\varphi(P*Q)=\varphi(P)\varphi(Q)\cdot q^{(ld-ke)/2}$$

Proof Let us write:

$$F(k,d):=q_1^{\frac{-k^2+kd+d+k}{2}}(q_1-1)^k\prod_{i=1}^k\frac{q_1^{i-1}-q_2}{q_1^i-1}$$

Then the LHS equals:

$$\mathrm{Sym}\left[P(z_1,...,z_k)Q(z_{k+1},...,z_{k+l})\prod_{1\le i\neq j\le k+l}\frac{z_i-q_1z_j}{z_i-z_j}\prod_{j>k+1}^{i\le k}\frac{(z_i-z_j)(z_i-qz_j)}{(z_i-q_1z_j)(z_i-q_2z_j)}\right]_{z_i=q_1^{-i}}\cdot$$

$$F(k+l,d+e)=\mathrm{Sym}\left[P(z_1,...,z_k)Q(z_{k+1},...,z_{k+l})\prod_{1\le i\neq j\le k}\frac{z_i-q_1z_j}{z_i-z_j}\prod_{k+1\le i\neq j\le k+l}\frac{z_i-q_1z_j}{z_i-z_j}\right.$$

$$\left.\prod_{j>k+1}^{i\le k}\frac{(z_i-qz_j)(z_j-q_1z_i)}{(z_i-q_2z_j)(z_j-z_i)}\right]_{z_i=q_1^{-i}}\cdot F(k+l,d+e)$$

The last factor in the numerator means that the only terms which do not vanish in the above Sym are those which put the variables $\{z_1,...,z_k\}$ before the variables $\{z_{k+1},...,z_{k+l}\}$, leaving the above equal to:

$$\mathrm{Sym}\left[P(z_1,...,z_k)\prod_{1\le i\neq j\le k}\frac{z_i-q_1z_j}{z_i-z_j}\right]_{z_i=q_1^{-i}}\mathrm{Sym}\left[Q(z_{k+1},...,z_{k+l})\prod_{k+1\le i\neq j\le k+l}\frac{z_i-q_1z_j}{z_i-z_j}\right]_{z_i=q_1^{-i}}$$

$$\prod_{k+1\le j\le k+l}^{1\le i\le k}\frac{(q_1^{-i}-qq_1^{-j})(q_1^{-j}-q_1^{1-i})}{(q_1^{-i}-q_2q_1^{-j})(q_1^{-j}-q_1^{-i})}\cdot F(k+l,d+e)=$$

$$=\varphi(P)\varphi(Q)q_1^{-ek+kl}\prod_{k+1\le j\le k+l}^{1\le i\le k}\frac{(q_1^{j-i-1}-q_2)(q_1^{j-i+1}-1)}{(q_1^{j-i}-q_2)(q_1^{j-i}-1)}\cdot\frac{F(k+l,d+e)}{F(k,d)F(l,e)}$$

[1] We could have just as well chosen q_2 instead of q_1, both maps work equally well

The products on the last line cancel out, leaving the desired RHS.

∎

3. Degrees and Coproducts

3.1. Given a shuffle element $P(z_1, ..., z_k) \in \mathcal{A}^+_{k,d}$ and a vector of integers:

$$s = (0 = s_0, s_1, ..., s_{k-1}, s_k = d) \in \mathbb{Z}^{k+1},$$

we can consider the limit:

$$\lim_{\xi \longrightarrow \infty} \frac{P(\xi z_1, ..., \xi z_i, z_{i+1}, ..., z_k)}{\xi^{s_i}} \tag{3.1}$$

If this limit exists for all $i \in \{0, ..., k\}$, then we say that the **degree** of P is $\leq s$. We will denote by $\mathcal{B}^s \subset \mathcal{A}^+_{k,d}$ the subspace of shuffle elements of degree $\leq s$. [2]

3.2. More explicitly, recall that any shuffle element can be written as:

$$P(z_1, ..., z_k) = \frac{\prod_{1 \leq a < b \leq k} (z_a - z_b)^2 \sum_\lambda c_\lambda m_\lambda(z_1, ..., z_k)}{\prod_{1 \leq a \neq b \leq k} (z_a - q_1 z_b)(z_a - q_2 z_b)} \tag{3.2}$$

where m_λ is the monomial symmetric function corresponding to the integer partition with k parts $\lambda = (\lambda_1 \geq ... \geq \lambda_k)$, and $c_\lambda \in \mathbb{K}$ are constants. It is easy to see that the limit (3.1) exists if and only if:

$$c_\lambda \neq 0 \Longrightarrow \lambda \leq \lambda^s, \qquad \text{where} \qquad \lambda^s_i = 2(n-i) + s_i - s_{i-1}$$

where the **dominance ordering** is defined for both degrees and partitions:

$$s \leq s' \qquad \Leftrightarrow \qquad s_k = s'_k \quad \text{and} \quad s_j \leq s'_j \quad \forall j$$

$$\lambda \leq \lambda' \qquad \Leftrightarrow \qquad |\lambda| = |\lambda'| \quad \text{and} \quad \lambda_1 + ... + \lambda_j \leq \lambda'_1 + ... + \lambda'_j \quad \forall j \tag{3.3}$$

3.3. With the above notations, if the limit (3.1) exists, then it equals:

$$\frac{\prod_{1 \leq a < b \leq i} (z_a - z_b)^2 \prod_{i < a < b \leq k} (z_a - z_b)^2 \sum'_\lambda c_\lambda m_{\lambda^{+i}}(z_1, ..., z_i) m_{\lambda^{-i}}(z_{i+1}, ..., z_k)}{q^{i(k-i)} \prod_{1 \leq a \neq b \leq i} (z_a - q_1 z_b)(z_a - q_2 z_b) \prod_{i < a \neq b \leq k} (z_a - q_1 z_b)(z_a - q_2 z_b)} \tag{3.4}$$

where $\sum'_\lambda$ goes over those partitions λ of the original sum which satisfy $\lambda_1 + ... + \lambda_i = \lambda^s_1 + ... + \lambda^s_i$, and $\lambda^{\pm i}$ denote the truncated partitions:

$$\lambda^{+i} = (\lambda_1 \geq ... \geq \lambda_i), \qquad \lambda^{-i} = (\lambda_{i+1} \geq ... \geq \lambda_k) \tag{3.5}$$

[2] The gradings k and d are implicitly contained in the vector s

3.4. Less explicitly, (3.4) implies that for any $P \in \mathcal{B}^s$ we can write:

$$\lim_{\xi \longrightarrow \infty} \frac{P(\xi z_1, ..., \xi z_i, z_{i+1}, ..., z_k)}{\xi^{s_i}} = \sum_a Q_{a,i}(z_1, ..., z_i) R_{a,i}(z_{i+1}, ..., z_k)$$

for certain rational functions $Q_{a,i}$, $R_{a,i}$, with the index a going over some finite set. Then we may define the map:

$$\Delta_i^s(P) = \sum_a Q_{a,i} \otimes R_{a,i} \tag{3.6}$$

Let us define the truncated degree vectors by analogy with (3.5):

$$s^{+i} = (0, s_1, ..., s_{i-1}, s_i) \qquad\qquad s^{-i} = (0, s_{i+1} - s_i, ..., s_{k-1} - s_i, s_k - s_i) \tag{3.7}$$

From the explicit description of Δ_i^s in (3.4), we can claim that:

$$\Delta_i^s : \mathcal{B}^s \longrightarrow \mathcal{B}^{s^{i+}} \otimes \mathcal{B}^{s^{i-}} \tag{3.8}$$

This entails the fact that $Q_{a,i}$ and $R_{a,i}$ are both shuffle elements and that they have the correct degree, which is easy to check.

3.5. In the next section, the maps Δ_i^s will be shown to induce coproducts on certain commutative subalgebras of $\mathcal{A}^+$. For the time being, it is easy to see that they satisfy "coassociativity", in the sense that:

$$(\Delta_i^{s^{j+}} \otimes 1) \circ \Delta_j^s = (1 \otimes \Delta_{j-i}^{s^{i-}}) \circ \Delta_i^s, \qquad \forall i < j \tag{3.9}$$

Indeed, this follows by applying formula (3.4) twice. Moreover, the following Lemma shows that the maps Δ_i^s "respect the multiplication":

Lemma 3.6. *Consider any degree vectors s and s', and shuffle elements $P_1 \in \mathcal{B}^s$ and $P_2 \in \mathcal{B}^{s'}$. Then $P_1 * P_2 \in \mathcal{B}^{s+s'}$, where:*

$$(s + s')_i = \max_{j+j'=i} s_j + s'_{j'} \tag{3.10}$$

Moreover, we have:

$$\Delta_i^{s+s'}(P_1 * P_2) = \sum_{\substack{i=j+j' \\ (s+s')_i = s_j + s'_{j'}}} \Delta_j^s(P_1) * \Delta_{j'}^{s'}(P_2) \tag{3.11}$$

Proof Let us assume $P_t \in \mathcal{A}_{k_t}^+$ and write $k = k_1 + k_2$. The LHS of (3.11) equals:

$$\lim_{\xi \longrightarrow \infty} \frac{(P_1 * P_2)(\xi z_1, ..., \xi z_i, z_{i+1}, ..., z_k)}{\xi^{(s+s')_i}} =$$

$$= \lim_{\xi \longrightarrow \infty} \sum_{\{1,...,k\}=A_1 \sqcup A_2}^{|A_1|=k_1, |A_2|=k_2} \frac{P_1(\xi^* z_{A_1}) P_2(\xi^* z_{A_2}) \prod_{x \in A_1} \prod_{y \in A_2} \omega\left(\frac{z_x}{z_y}\right)}{\xi^{(s+s')_i}}$$

where the exponent $*$ is 1 or 0, depending on whether or not the corresponding subscript of z lies in $\{1,...,i\}$ or not. The above quantity equals:

$$\lim_{\xi \longrightarrow \infty} \sum_{i=j+j'} \sum_{\substack{\{1,...,i\}=B_1 \sqcup B_2 \\ \{i+1,...,k\}=C_1 \sqcup C_2}}^{\substack{|B_1|=j, |B_2|=j' \\ |C_1|=k_1-j, |C_2|=k_2-j'}} \frac{P_1(\xi z_{B_1}, z_{C_1}) P_2(\xi z_{B_2}, z_{C_2}) \prod_{y \in B_2 \cup C_2}^{x \in B_1 \cup C_1} \omega\left(\frac{z_x}{z_y}\right)}{\xi^{(s+s')_i}}$$

By using the fact that $\omega(0) = \omega(\infty) = 1$, the above limit is finite, and $P_1 * P_2$ therefore lies in $\mathcal{B}^{s+s'}$. Moreover, we can write the above as:

$$\lim_{\xi \longrightarrow \infty} \sum_{i=j+j'} \sum_{\substack{\{1,...,i\}=B_1 \sqcup B_2 \\ \{i+1,...,k\}=C_1 \sqcup C_2}}^{\substack{|B_1|=j, |B_2|=j' \\ |C_1|=k_1-j, |C_2|=k_2-j'}} \frac{P_1(\xi z_{B_1}, z_{C_1}) P_2(\xi z_{B_2}, z_{C_2}) \prod_{y \in B_2}^{x \in B_1} \omega\left(\frac{z_x}{z_y}\right) \prod_{y \in C_2}^{x \in C_2} \omega\left(\frac{z_x}{z_y}\right)}{\xi^{\max_{i=j+j'} s_j + s'_{j'}}}$$

which is easily seen to be the RHS of the desired relation.

□

4. Minimal shuffle elements

4.1. It is quite difficult to express the wheel conditions (2.6) explicitly. But the notion of degree of the previous section allows us to assert the following:

Lemma 4.2. *Consider the degree vector:*

$$s(k,d)_i = \left\lceil \frac{id}{k} \right\rceil - 1 + \delta_i^k, \qquad i \in \{1,...,k\} \tag{4.1}$$

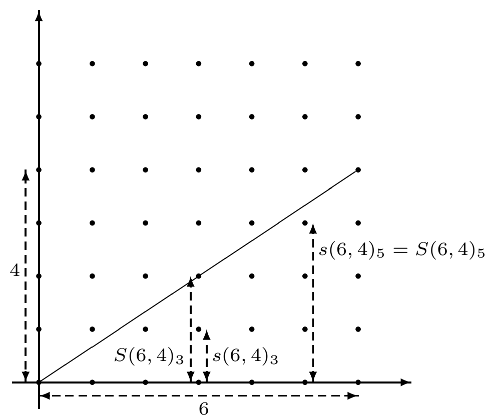
Then $\dim \mathcal{B}^{s(k,d)} = 1$.

Lemma 4.3. *Consider the degree vector:*

$$S(k,d)_i = \left\lfloor \frac{id}{k} \right\rfloor, \qquad i \in \{1,...,k\} \tag{4.2}$$

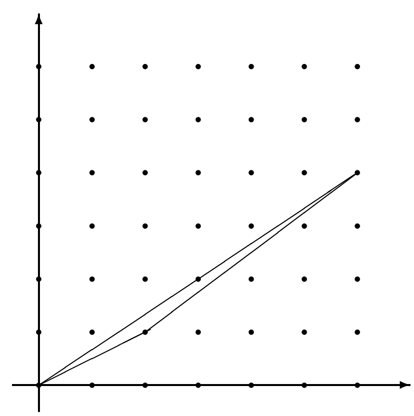
Then $\dim \mathcal{B}^{S(k,d)} = p(\gcd(k,d))$*, where* $p(n)$ *denotes the number of partitions of the number* n*.*

Remark 4.4. The above numbers have a geometric interpretation. If we consider (k,d) to be a lattice point, then $s(k,d)_i$ (respectively, $S(k,d)_i$) is the highest $y-$coordinate of a lattice point on the vertical line $x = i$ which is strictly below (respectively, on or below) the line $(0,0),(k,d)$.



In particular, when $\gcd(k,d) = 1$ the two lemmas assert the exact same thing.

Remark 4.5. The above interpretation is closely connected to the following geometric setup, which will be very important to us in Section 6. Consider triangles with vertices $(0,0),(i,j),(k,d)$, for some $i \in \{1,...,k-1\}$ and $j = s(k,d)_i$:



We will only look at triangles with no inside lattice points. Such a triangle is called **minimal** if it has no lattice points on the edges, except the vertices. Such a triangle is pictured above. If it has lattice points on only one edge, it will be called **semi-minimal**. We will never consider triangles with lattice points on ≥ 2 edges. Note that for any (k,d) with $k > 1$, there exists such a minimal triangle for some i, simply by taking one of minimal area.

4.6. In [1], the authors have proved the case $d = 0$ of Lemma 4.3. We will use a slightly modified version of their argument in order to prove half of Lemmas 4.2 and 4.3, namely the fact that the required dimensions are $\leq$ than what we claim they are. For any degree vector $s = (0 = s_0, s_1, ..., s_{k-1}, s_k = d)$ and any partition $\mu = (\mu_1 \geq ... \geq \mu_t)$ of k, consider the evaluation map:

$$\varphi_\mu : \mathcal{B}^s \longrightarrow \mathbb{K}[y_1^{\pm 1}, ..., y_t^{\pm 1}],$$

$$\varphi_\mu(P) = p(z_1, ..., z_k)|_{z_{\mu_1+...+\mu_{s-1}+x}=y_s q_1^x}, \quad \forall s \forall 1 \leq x \leq \mu_s$$

where p is the Laurent polynomial of (2.5). Note that $\varphi_{(k)}$ is, up to a constant, simply the map of Section (2.7). This construction gives rise to the subspaces:

$$\mathcal{B}^s \supset \mathcal{B}^s_\mu = \bigcap_{\nu > \mu} \ker \varphi_\nu \tag{4.3}$$

where $>$ refers to the dominance ordering of (3.3). It is easy to see that these subspaces form a filtration of $\mathcal{B}^s$ (if we set $\mathcal{B}^s_{(k)} = \mathcal{B}^s$), since:

$$\mu \leq \mu' \Longrightarrow \mathcal{B}^s_\mu \subset \mathcal{B}^s_{\mu'}$$

Proof of the $\leq$ half of Lemma 4.2 and Lemma 4.3: We will now prove the inequalities:

$$\dim \mathcal{B}^s \leq 1, \qquad \dim \mathcal{B}^S \leq p(\gcd(k, d))$$

where $s = s(k, d)$ and $S = S(k, d)$ are the degree vectors of (4.1) and (4.2). By using the filtration (4.3), it is easy to see that these inequalities are equivalent to:

$$\dim \varphi_\mu(\mathcal{B}^s_\mu) \leq \delta^{(k)}_\mu, \qquad \dim \varphi_\mu(\mathcal{B}^S_\mu) \leq \sum_{|\rho| = \gcd(k,d)} \delta_\mu^{\overline{\frac{k}{\gcd(k,d)}} \cdot \rho}, \qquad \forall |\mu| = k \tag{4.4}$$

where for a partition $\rho = (\rho_1 \geq \rho_2 \geq ...)$ and a positive integer n, we write $n \cdot \rho = (n\rho_1 \geq n\rho_2 \geq ...)$. Let $\sigma \in \{s, S\}$, take a shuffle element $P \in \mathcal{B}^\sigma_\mu$ and let $r = \varphi_\mu(P)$. Because P satisfies the wheel condition (2.6), the Laurent polynomial r vanishes for:

$$y_j = q_2 q^{a-b} y_i, \quad a \in \{0, 1, ..., \mu_i - 2\}, \quad b \in \{0, ..., \mu_j - 1\} \tag{4.5}$$

$$y_j = q_1 q^{a-b} y_i, \quad a \in \{0, 1, ..., \mu_i - 2\}, \quad b \in \{0, ..., \mu_j - 1\} \tag{4.6}$$

for $i < j$, with the correct multiplicities. Because P lies in $\mathcal{B}^\sigma_\mu = \bigcap_{\nu > \mu} \ker \varphi_\nu$, we see that r also vanishes for:

$$y_j = q_1^{\mu_i - b} y_i \quad \text{and} \quad y_j = q_1^{-b} y_i, \qquad b \in \{0, 1, ..., \mu_j - 1\} \tag{4.7}$$

for $i < j$. Therefore, the Laurent polynomial r is divisible by:

$$r_0(y_1, ..., y_t) = \prod_{1 \leq i < j \leq k} \left[\prod_{b=0}^{\mu_j - 1} (y_j - q_1^{\mu_i - b} y_i)(y_j - q_1^{-k} y_i) \right.$$

$$\left. \prod_{b=0}^{\mu_j - 1} \prod_{a=0}^{\mu_i - 2} (y_j - q_2 q^{a-b} y_i)(y_j - q_1 q^{a-b} y_i) \right] \tag{4.8}$$

This polynomial has total degree:

$$\deg(r_0) = \sum_{i<j} 2\mu_i\mu_j = k^2 - \sum_i \mu_i^2$$

and degree at most:

$$\deg_{y_i}(r_0) = \sum_{i \neq j} 2\mu_i\mu_j = 2k\mu_i - \mu_i^2$$

in each variable y_i. As for r, it's easy to see that it has total degree:

$$\deg(r) = k(k-1) + d$$

Because the degree of P is $\leq \sigma$, then it has degree at most:

$$\deg_{y_i}(r) = 2k\mu_i - \mu_i(\mu_i + 1) + \sigma_{\mu_i}$$

in each variable y_i. So the quotient r/r_0 is a Laurent polynomial of total degree:

$$\sum_i \mu_i(\mu_i - 1) + d$$

and degree at most:

$$\mu_i(\mu_i - 1) + \sigma_{\mu_i}$$

in each variable y_i. If $\sum_{i=1}^t \sigma_{\mu_i} < d$, then $r/r_0 = 0$ and therefore $\dim \varphi_\mu(\mathcal{B}_\mu^\sigma) = 0$. If $\sum_{i=1}^t \sigma_{\mu_i} = d$, then $r/r_0 = \text{const} \cdot y_1^{\sigma_{\mu_1}} ... y_t^{\sigma_{\mu_t}}$ and therefore $\dim \varphi_\mu(\mathcal{B}_\mu^\sigma) \leq 1$. So looking at the definition of s in (4.1), the first inequality of (4.4) follows from the relation:

$$\sum_{i=1}^t \mu_i = k \Longrightarrow \sum_{i=1}^t \left\lceil \frac{\mu_i d}{k} \right\rceil - 1 + \delta_{\mu_i}^k \leq d$$

with equality if and only if $\mu = (k)$. In the case of S from (4.2), the second inequality of (4.4) follows from:

$$\sum_{i=1}^t \mu_i = k \Longrightarrow \sum_{i=1}^t \left\lfloor \frac{\mu_i d}{k} \right\rfloor \leq d$$

with equality if and only if $k/\gcd(k,d)$ divides μ_i for all i. Both these inequalities are elementary.

□

4.7. Take a pair of integers (a,b) with $\gcd(a,b) = 1$ and look at:

$$\mathcal{C}^{(a,b)} = \bigoplus_{n \geq 0} \mathcal{B}^{S(na,nb)} \subset \bigoplus_{n \geq 0} \mathcal{A}^+_{na,nb}$$

where recall that $S(na,nb)_i = \left\lfloor \frac{ia}{b} \right\rfloor$. We will often abbreviate this subspace by $\mathcal{C}$ (when a,b are clear from context), and write $\mathcal{C}_n$ for its graded parts.

Lemma 4.8. *The vector space $\mathcal{C}$ is a subalgebra of $\mathcal{A}^+$, and the map:*

$$\Delta : \mathcal{C}_n \longrightarrow \bigoplus_{i=0}^{n} \mathcal{C}_i \otimes \mathcal{C}_{n-i}, \qquad \Delta(P) = \sum_{i=0}^{n} \Delta_{ia}^{S(na,nb)}(P)$$

is a coproduct that respects multiplication.

Proof The fact that Δ_{ia} takes values in $\mathcal{C}_i \otimes \mathcal{C}_{n-i}$ follows from (3.8), coassociativity follows from (3.9), and the fact that Δ respects the multiplication follows from Lemma 3.6.

□

4.9. The subspace $\mathcal{B}^{s(na,nb)} \subset \mathcal{B}^{S(na,nb)} = \mathcal{C}_n$ consists precisely of those shuffle elements P which satisfy:

$$\Delta(P) = P \otimes 1 + 1 \otimes P \tag{4.9}$$

We know that this subspace has dimension at most 1, but we have yet to show that it is non-empty. This will be done in Subsection 5.7, where we will explicitly construct a shuffle element $P_n \in \mathcal{B}^{s(na,nb)}$ such that:

$$\varphi(P_n) = \alpha_1 \cdot \frac{(1-q^{-1})^{-k}}{q_1^{\frac{n}{2}} - q_1^{-\frac{n}{2}}} \tag{4.10}$$

where we write:

$$\alpha_n = \frac{(q_1^n - 1)(q_2^n - 1)(q^{-n} - 1)}{n} \in \mathbb{K} \tag{4.11}$$

This shuffle element is called **minimal**. When we wish to emphasize that it has bidegrees (na, nb), we will denote it by $P_{na,nb}$, and in Subsection 5.7 we will show that it is given by:

$$P_{na,nb} = (1-q^{-1})^{1-na} \frac{(q_1 - 1)(q_2 - 1)}{(q_1^n - 1)(q_2^n - 1)}$$

$$\sum_{x=0}^{n-1} q^x \cdot \mathrm{Sym} \left[\frac{z_1 \prod_{j=1}^{na} z_j^{\lfloor \frac{jd}{k} \rfloor - \lfloor \frac{(j-1)d}{k} \rfloor} \cdot z_{(n-x)a+1} \dots z_{(n-1)a+1}}{z_k \left(\frac{z_1}{z_2} - q\right) \dots \left(\frac{z_{k-1}}{z_k} - q\right) \cdot z_{(n-x)a} \dots z_{(n-1)a}} \prod_{i<j} \omega\left(\frac{z_i}{z_j}\right) \right] \tag{4.12}$$

4.10. For the time being, we will prove the following:

Proposition 4.11. *For any m, n, the elements P_n and P_m commute.*

Proof Since P_n and P_m satisfy (4.9), and Δ respects the multiplication, we have:

$$\Delta([P_n, P_m]) = [P_n, P_m] \otimes 1 + 1 \otimes [P_n, P_m]$$

As was mentioned before, this implies that $[P_n, P_m]$ lies in the space $\mathcal{B}^{s((m+n)a,(m+n)b)}$, which we know has dimension ≤ 1. From the proof of Lemma

4.2, we know that if this space is non-zero, then it it does not lie in the kernel of φ. By Proposition 2.8, we have that $\varphi([P_n, P_m]) = \varphi(P_n)\varphi(P_m) - \varphi(P_m)\varphi(P_n) = 0$, and this therefore implies that $[P_n, P_m] = 0$.

□

4.12. Still keeping a, b fixed, we can define new shuffle elements $Q_n \in \mathcal{C}_n$ via the following generating series:

$$\sum_{n=0}^{\infty} x^n Q_n = \exp\left(\sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha_1} x^n P_n\right) \tag{4.13}$$

where α_n are the constants defined in (4.11). Just as the P_n, the elements Q_n will all commute among themselves. From (4.13), it is easy to see that:

$$\Delta(Q_n) = \sum_{i=0}^{n} Q_i \otimes Q_{n-i} \tag{4.14}$$

where we write $Q_0 = 1$. The map φ restricted to $\mathcal{C}$ is multiplicative, so applying it to (4.13) gives us:

$$\varphi(Q_n) = \frac{q_1^{-\frac{n}{2}} - q_1^{\frac{n}{2}}}{(1-q^{-1})^k} \cdot \frac{(q_2 - 1)(q_1 - q_2^{-1})}{q_1 - 1} \tag{4.15}$$

When we will want to emphasize the fact that Q_n has bidegrees (na, nb), we will denote it by $Q_{na,nb}$. In Subsection 5.7, it will be shown that:

$$Q_{na,nb} = (1-q^{-1})^{1-na}.$$

$$\sum_{x=0}^{n-1} \mathrm{Sym}\left[\frac{z_1 \prod_{j=1}^{na} z_j^{\left\lfloor \frac{jd}{k} \right\rfloor - \left\lfloor \frac{(j-1)d}{k} \right\rfloor} \cdot z_{(n-x)a+1} \dots z_{(n-1)a+1}}{z_k \left(\frac{z_1}{z_2} - q\right) \dots \left(\frac{z_{k-1}}{z_k} - q\right) \cdot z_{(n-x)a} \dots z_{(n-1)a}} \prod_{i<j} \omega\left(\frac{z_i}{z_j}\right)\right] \tag{4.16}$$

4.13. For the time being, we will prove the following:

Proposition 4.14. *The relations* (4.14) *and* (4.15) *uniquely determine the shuffle elements $Q_n \in \mathcal{C}_n$, recursively in n.*

Proof Suppose that we have already uniquely found $Q_1, ..., Q_{n-1}$ and there are two elements $Q \neq Q' \in \mathcal{C}_n$ which both have the same coproduct. It then follows that:

$$\Delta(Q - Q') = (Q - Q') \otimes 1 + 1 \otimes (Q - Q')$$

The above property is equivalent to $Q - Q' \in \mathcal{B}^{s(na,nb)}$. However, $\varphi(Q - Q') = 0$, and by the same token as in the proof of Proposition 4.11, it follows that $Q - Q' = 0$.

□

5. Explicit formulas

5.1. Conjecture 2.6 implies that any shuffle element can be written (non-uniquely) as a linear combination of [3]:

$$Y_{k,d}^{\lambda_1,...,\lambda_k} = z_1^{\lambda_1} * ... * z_1^{\lambda_k} = \text{Sym}\left[z_1^{\lambda_1}...z_k^{\lambda_k} \prod_{1\leq i<j\leq k} \omega\left(\frac{z_i}{z_j}\right)\right]$$

as $\lambda_1, ..., \lambda_k$ go over the integers. As it will be shown in [2], there is another set of elements that are perhaps more important:

$$X_{k,d}^{\lambda_1,...,\lambda_k} = \text{Sym}\left[\frac{z_1^{\lambda_1}...z_k^{\lambda_k}}{\left(\frac{z_1}{z_2}-q\right)...\left(\frac{z_{k-1}}{z_k}-q\right)} \prod_{1\leq i<j\leq k} \omega\left(\frac{z_i}{z_j}\right)\right]$$

Proposition 5.2. *For any $\lambda_1, ..., \lambda_k \in \mathbb{Z}$, we have $X_{k,d}^{\lambda_1,...,\lambda_k} \in \mathcal{A}^+_{k,\lambda_1+...+\lambda_k}$.*

Proof It is easy to see that the rational function $X_{k,d}^{\lambda_1,...,\lambda_k}$ has the appropriate poles and vanishes when $z_i = z_j$. Since it is symmetric, it must therefore be divisible by $(z_i - z_j)^2$. It is quite easy to see that each summand of the above Sym vanishes when any three of the variables are set to $(1, q_1, q)$ or $(1, q_2, q)$, and therefore $X_{k,d}^{\lambda_1,...,\lambda_k}$ is a shuffle element. In fact, the denominator $\left(\frac{z_1}{z_2}-q\right)...\left(\frac{z_{k-1}}{z_k}-q\right)$ is maximal so that expressions like the above still satisfy the wheel conditions.

□

5.3. Let us fix a bidegree $(k, d) = (na, nb)$ for $\gcd(a, b) = 1$. For any binary string $\varepsilon = (\varepsilon_1, ..., \varepsilon_{n-1}) \in \{0, 1\}^{n-1}$, consider the degree vector:

$$S(k,d)_i^{\varepsilon} = \left\lfloor \frac{id}{k} \right\rfloor - \varepsilon_{i/a}$$

Naturally, the term $\varepsilon_{i/a}$ only appears when $a|i$. We have:

$$S(k,d)^{(0,...,0)} = S(k,d) \qquad\qquad S(k,d)^{(1,...,1)} = s(k,d)$$

and the general $S(k, d)^{\varepsilon}$ simply interpolates between these two extremes. As there will be no source of confusion about k and d, we will often denote this degree vector simply by S^{ε}.

Proposition 5.4. *For any vector ε as above, the shuffle element:*

$$X_{k,d}^{\varepsilon} := X_{k,d}^{1+S_1^{\varepsilon}, S_2^{\varepsilon}-S_1^{\varepsilon},...,S_{k-1}^{\varepsilon}-S_{k-2}^{\varepsilon}, S_k^{\varepsilon}-S_{k-1}^{\varepsilon}-1}$$

lies in $\mathcal{B}^{S(k,d)}$. Moreover,

$$\varphi(X_{k,d}^{\varepsilon}) = q_1^{\frac{n}{2}-\#\text{ of ones in }\varepsilon}(1-q_2) \tag{5.1}$$

and:

[3] If we do not accept the Conjecture, then $\mathcal{A}$ must be replaced by $\tilde{\mathcal{A}}$ in all our statements

$$\Delta(X^{\varepsilon}_{k,d}) = \sum_{t\geq 1}(-q)^{-t+\delta^0_{x_1}} \sum_{\substack{u_1,...,u_t\in\varepsilon^{-1}(1)\\ v_1,...,v_t\in\varepsilon^{-1}(0)}}^{0\leq u_1<v_1<...<u_t<v_t\leq n} \tag{5.2}$$

$$X^{(\varepsilon_{u_1+1},...,\varepsilon_{v_1-1})}_{(v_1-u_1)a,(v_1-u_1)b} * ... * X^{(\varepsilon_{u_t+1},...,\varepsilon_{v_t-1})}_{(v_t-u_t)a,(v_t-u_t)b} \otimes X^{(\varepsilon_1,...,\varepsilon_{u_1})}_{u_1a,u_1b} * ... * X^{(\varepsilon_{v_t+1},...,\varepsilon_{n-1})}_{(n-v_t)a,(n-v_t)b}$$

where $*$ denotes the shuffle product. In the above sum, we make the convention that $\varepsilon_0 = 1$ and $\varepsilon_n = 0$.

Proof Explicitly, we have:

$$X^{\varepsilon}_{k,d} = \sum_{\sigma\in S(k)} \frac{z_{\sigma(1)}\prod_{j=1}^k z_{\sigma(j)}^{S^{\varepsilon}_j - S^{\varepsilon}_{j-1}}}{z_{\sigma(k)}\left(\frac{z_{\sigma(1)}}{z_{\sigma(2)}}-q\right)...\left(\frac{z_{\sigma(k-1)}}{z_{\sigma(k)}}-q\right)} \prod_{i<j}\omega\left(\frac{z_{\sigma(i)}}{z_{\sigma(j)}}\right) \tag{5.3}$$

To show that the above sum lies in $\mathcal{B}^{S(k,d)}$, we need to multiply the variables $z_1,...,z_i$ by ξ and show that we get something of order no greater than $S(k,d)_i$ as $\xi\to\infty$. In fact, we will show that each summand of (5.3) has this property. The ω's stay finite, since $\omega(0)=\omega(\infty)=1$, so they will not bother us. A permutation $\sigma\in S(k)$ is determined by:

$$A = \sigma^{-1}(\{1,...,i\}) \qquad \text{and} \qquad \sigma'\in S(A), \quad \sigma''\in S(\overline{A})$$

where $\overline{A} = \{1,...,k\}\backslash A$. The set A will be the most important part of the data; it will be of the form:

$$A = \{x_1+1, y_1\}\cup\{x_2+1,y_2\}\cup ...\{x_t+1,y_t\} \tag{5.4}$$

where

$$x_1<y_1<x_2<y_2<...<x_t<y_t\in\{0,...,k\}$$

are certain indices such that:

$$\sum_{j=1}^{y}(y_j - x_j) = i \tag{5.5}$$

The term corresponding to σ in (5.3) gets a contribution of ξ to the power at most $\sum_{j=1}^t S^{\varepsilon}_{y_t} - S^{\varepsilon}_{x_t} + \delta^0_{x_1} - \delta^k_{y_t}$ from the numerator, and ξ to the power exactly $-t+\delta^k_{y_t}$ from the denominator. Therefore, the fact that each summand of $X^{\varepsilon}_{k,d}$ in (5.3) has the required degree is equivalent to:

$$\sum_{j=1}^{t}\left(S^{\varepsilon}_{y_j} - S^{\varepsilon}_{x_j}\right) - t + \delta^0_{x_1} \leq S(k,d)_i = \left\lfloor \frac{id}{k}\right\rfloor \tag{5.6}$$

The above follows from (5.5) and the simpler inequality:

$$S^\varepsilon_y - S^\varepsilon_x - 1 + \delta^0_x = \left\lfloor \frac{yd}{k} \right\rfloor - \left\lfloor \frac{xd}{k} \right\rfloor - \varepsilon_{y/a} + \varepsilon_{x/a} - 1 \leq \left\lfloor \frac{(y-x)d}{k} \right\rfloor \tag{5.7}$$

which holds for any $x < y \in \{0, ..., k\}$. To compute the image of $X^\varepsilon_{k,d}$ under Δ, we need to take those terms of top degree, i.e. the ones which give equality in (5.6). By (5.7), this happens only when $x_j = u_j a$ for $\varepsilon_{u_j} = 1$ and $y_j = v_j a$ for $\varepsilon_{v_j} = 0$, for all $j \leq t$. The corresponding term, recalling that $\omega(0) = \omega(\infty) = 1$, equals precisely (5.2).

To prove (5.1), we need to look back at (5.3) and the definition of φ:

$$\varphi(X^\varepsilon_{k,d}) = q_1^{\frac{-k^2+kd+k+d}{2}} (q_1-1)^k \prod_{i=1}^k \frac{q_1^{i-1} - q_2}{q_1^i - 1} \sum_{\sigma \in S(k)} \frac{z_{\sigma(1)} \prod_{j=1}^k z_{\sigma(j)}^{S^\varepsilon_j - S^\varepsilon_{j-1}}}{z_{\sigma(k)} \left(\frac{z_{\sigma(1)}}{z_{\sigma(2)}} - q\right) ... \left(\frac{z_{\sigma(k-1)}}{z_{\sigma(k)}} - q\right)}$$

$$\prod_{i<j} \frac{(z_{\sigma(j)} - q_1 z_{\sigma(i)})(z_{\sigma(i)} - q z_{\sigma(j)})}{(z_{\sigma(j)} - z_{\sigma(i)})(z_{\sigma(i)} - q_2 z_{\sigma(j)})} \Big|_{z_i = q_1^{-i}}$$

Only one term survives when we evaluate the above at $z_i = q_1^{-i}$, namely the one corresponding to the identity permutation. Therefore, the above gives:

$$\varphi(X^\varepsilon_{k,d}) = q_1^{\frac{-k^2+kd+k+d}{2}} (q_1-1)^k \frac{q_1^{\sum_{j=1}^k j(S^\varepsilon_{j-1} - S^\varepsilon_j)}}{(1-q_2)^{k-1}} \prod_{i=1}^k \frac{q_1^{i-1} - q_2}{q_1^i - 1} \prod_{i<j} \frac{(q_1^{-j} - q_1^{1-i})(q_1^{-i} - q_2 q_1^{1-j})}{(q_1^{-j} - q_1^{-i})(q_1^{-i} - q_2 q_1^{-j})}$$

$$= q_1^{\frac{-kd+d+k}{2} + \sum_{j=1}^{k-1} S^\varepsilon_j} (1-q_2) = q_1^{\frac{n}{2} - \#\text{ of ones in } \varepsilon} (1-q_2),$$

where the last equality follows by Pick's theorem.

□

5.5. Certain particularly important cases of the above construction are:

$$X^{(0^a 1^b)}_{k,d} = X_{k,d}^{(\overbrace{0,...,0}^{a \text{ ones}}, \overbrace{1,...,1}^{b \text{ zeroes}})}$$

Proposition 5.6. *For any $(k,d) = (na, nb)$ with $\gcd(a,b) = 1$, we have:*

$$\sum_{x+y=n-2}^{x,y\geq 0} q^y \cdot X^{(0^x)} * X^{(1^y)} = X^{(0^{n-1})} - q^{n-1} X^{(1^{n-1})} \tag{5.8}$$

$$\sum_{x+y=n-2}^{x,y\geq 0} X^{(0^x)} * X^{(1^y)} = X^{(0^{n-1})} - q X^{(1^{n-1})} + (1-q) \sum_{x+y=n-1}^{x,y>0} X^{(0^x 1^y)} \tag{5.9}$$

where $X^{(0^x 1^y)}$ denotes $X^{(0^x 1^y)}_{(x+y+1)a,(x+y+1)b}$.

Proof Let $S = S(k,d)$. By definition, for $x, y \geq 0$, we have:

$$X^{(0^x)} * X^{(1^y)} =$$

$$= \mathrm{Sym}\left[\frac{z_1 \prod_{j=1}^{(x+y+2)a} z_j^{S_j - S_{j-1}} \cdot \frac{z_{(x+1)a+1} \cdots z_{(x+y+1)a+1}}{z_{(x+1)a} \cdots z_{(x+y+1)a}}}{z_k \left(\frac{z_1}{z_2} - q\right) \cdots \left(\frac{z_{(x+1)a-1}}{z_{(x+1)a}} - q\right)\left(\frac{z_{(x+1)a+1}}{z_{(x+1)a+2}} - q\right) \cdots \left(\frac{z_{k-1}}{z_k} - q\right)} \prod_{i<j} \omega\left(\frac{z_i}{z_j}\right)\right] =$$

$$= \mathrm{Sym}\left[\frac{z_1 \prod_{j=1}^{(x+y+2)a} z_j^{S_j - S_{j-1}} \cdot \frac{z_{(x+1)a+1} \cdots z_{(x+y+1)a+1}}{z_{(x+1)a} \cdots z_{(x+y+1)a}} \left(\frac{z_{(x+1)a}}{z_{(x+1)a+1}} - q\right)}{z_k \left(\frac{z_1}{z_2} - q\right) \cdots \left(\frac{z_{k-1}}{z_k} - q\right)} \prod_{i<j} \omega\left(\frac{z_i}{z_j}\right)\right] =$$

$$= X^{(0^{x+1}1^y)} - qX^{(0^x 1^{y+1})}.$$

The desired relations all follow by adding up these identites.

□

5.7. To any pair $(k,d) = (na, nb)$ with $\gcd(a,b) = 1$, sections 4.9 and 4.12 have claimed the existence of shuffle elements:

$$P_{k,d} := P_n \subset \mathcal{B}^{s(k,d)} \subset \mathcal{A}_k^+, \qquad Q_{k,d} := Q_n \subset \mathcal{B}^{S(k,d)} \subset \mathcal{A}_k^+$$

These are determined by (4.9) and (4.10), respectively (4.14) and (4.15). In this section, we will prove that they exist and are given by formulas (4.12) and (4.16). In our new notation, these two relations claim that:

$$P_{k,d} = (1 - q^{-1})^{1-k} \frac{(q_1 - 1)(q_2 - 1)}{(q_1^n - 1)(q_2^n - 1)} \sum_{x+y=n-1}^{x,y \geq 0} q^y X_{k,d}^{(0^x 1^y)} \tag{5.10}$$

$$Q_{k,d} = (1 - q^{-1})^{1-k} \sum_{x+y=n-1}^{x,y \geq 0} X_{k,d}^{(0^x 1^y)} \tag{5.11}$$

To prove this, we need to show that the right hand sides of the above relations verify properties (4.9) (respectively (4.14)) and (4.10) (respectively (4.15)). The latter two relations are simple consequences of (5.1), so let us prove the former. We have, by (5.2):

$$\Delta\left(\sum_{x+y=n-1}^{x,y \geq 0} q^y X_{k,d}^{(0^x 1^y)}\right) = 1 \otimes \left(\sum_{x+y=n-1}^{x,y \geq 0} q^y X_{k,d}^{(0^x 1^y)}\right) + \left(\sum_{x+y=n-1}^{x,y \geq 0} q^y X_{k,d}^{(0^x 1^y)}\right) \otimes 1 -$$

$$- \sum_{\substack{x,y,w,z \geq 0 \\ x+y+z+w=n-3}} q^y X^{(0^x)} * X^{(1^y)} \otimes q^w X^{(0^z 1^w)}$$

$$+ \sum_{\substack{x,y,w \geq 0 \\ x+y+z=n-2}} X^{(0^x)} \otimes q^w X^{(0^z 1^w)} - \sum_{\substack{x,y,z \geq 0 \\ x+y+z=n-2}} q^y X^{(1^y)} \otimes q^w X^{(0^z 1^w)}$$

where $X^{(0^{-1})} := 1$ and we omit the subscripts [4] under the X's for convenience. Relation (5.8) precisely says that the last two lines in the above cancel out, so this proves that our $P_{k,d}$ satisfies (4.9). As for (4.14), we have by definition:

$$\Delta\left(\sum_{x+y=n-1}^{x,y\geq 0} X_{k,d}^{(0^x1^y)}\right) = 1\otimes\left(\sum_{x+y=n-1}^{x,y\geq 0} X_{k,d}^{(0^x1^y)}\right) + \left(\sum_{x+y=n-1}^{x,y\geq 0} X_{k,d}^{(0^x1^y)}\right)\otimes 1 -$$

$$-q^{-1}\sum_{\substack{x,y,w,z\geq 0\\ x+y+z+w=n-3}} X^{(0^x)} * X^{(1^y)} \otimes X^{(0^z1^w)}$$

$$+\sum_{\substack{x,y,w\geq 0\\ x+y+z=n-2}} X^{(0^x)} \otimes X^{(0^z1^w)} - q^{-1}\sum_{\substack{x,y,z\geq 0\\ x+y+z=n-2}} X^{(1^y)} \otimes X^{(0^z1^w)}$$

We will omit the subscripts under the X's for convenience. By applying (5.9), the above gives rise to:

$$\Delta\left(\sum_{x+y=n-1}^{x,y\geq 0} X_{k,d}^{(0^x1^y)}\right) = 1\otimes\left(\sum_{x+y=n-1}^{x,y\geq 0} X_{k,d}^{(0^x1^y)}\right) +$$

$$+\left(\sum_{x+y=n-1}^{x,y\geq 0} X_{k,d}^{(0^x1^y)}\right)\otimes 1 + (1-q^{-1})\sum_{\substack{x,y>0,\quad z,w\geq 0\\ x+y+z+w=n-2}} X^{(0^x1^y)} \otimes X^{(0^z1^w)}$$

This precisely states that $Q_{k,d}$ verifies property (4.14), as desired.

5.8. The existence on the non-zero shuffle element $P_{k,d}$ of (4.12) completes the proof of Lemma 4.2. We will now proceed to the proof of Lemma 4.3. Consider the commutative algebra of symmetric polynomials in infinitely many variables:

$$\Lambda = \mathbb{K}[x_1, x_2, ...]^{\mathrm{Sym}}$$

It has several important systems of generators, for instance:

$$p_n = \sum_{i=1}^{\infty} x_i^n \qquad \text{and} \qquad h_n = \sum_{n_1+n_2+...=n} x_1^{n_1} x_2^{n_2} ...$$

in the sense that $\Lambda = \mathbb{K}[p_1, p_2, ...] = \mathbb{K}[h_1, h_2, ...]$. A linear basis of Λ is given by the monomial symmetric functions:

$$m_\lambda(x_1, x_2, ...) = \sum_{\sigma\in S(\infty)} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} ...$$

[4]The subscripts are all equal to $\left(\frac{rk}{n}, \frac{rd}{n}\right)$, where r is 1 plus the length of the vector in the superscript of the corresponding X

5.9. The algebra Λ is graded by $\deg p_n = n$, and it comes with a natural coproduct induced from the plethystic multiplication of partitions:

$$\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n, \qquad \Delta(h_n) = \sum_{i=0}^{n} h_i \otimes h_{n-i}$$

$$\Delta(m_\lambda) = \sum_{\lambda = \mu \sqcup \nu} m_\mu \otimes m_\nu,$$

where the union means joining partitions. The remainder of Lemma 4.3 is a consequence of the following result:

Theorem 5.10. *For any (a, b) with $\gcd(a, b) = 1$, the map:*

$$\Psi(p_n) = P_{na,nb}$$

extends to an isomorphism of $\mathbb{K}-$algebras:

$$\Psi : \Lambda \cong \mathcal{C}^{a,b} \tag{5.12}$$

which respects Δ. In particular, the degree n piece $\mathcal{C}^{a,b}_n = \mathcal{B}^{S(na,nb)}$ has dimension exactly equal to $p(n)$.

Proof By Proposition 4.11, the map Ψ is well-defined. By (4.9) and (4.10), it respects Δ. The dimension of Λ_n is equal to $p(n)$, the number of partitions of n. By what we proved so far concerning Lemma 4.3, the dimension of $\mathcal{C}_n$ is at most equal to $p(n)$. Therefore, to prove that Ψ is an isomorphism, it suffices to prove that it is injective. To this end, suppose that we have a non-trivial relation:

$$\sum_{|\lambda|=n} c_\lambda \Psi(m_\lambda) = 0$$

for some constants $c_\lambda \in \mathbb{K}$, and take $n \in \mathbb{N}$ minimal with this property. Also, let $m \in \{1, ..., n\}$ be minimal with the property that some $\lambda^0 = (\lambda^0_1 \geq ... \geq n')$ appears with $c_{\lambda^0} \neq 0$ in the above relation. We have $n' < n$, because we already know $\Psi(m_{(n)}) = \Psi(p_n) \neq 0$. Apply the coproduct to the above relation:

$$\sum_{|\mu|+|\nu|=n} c_{\mu \sqcup \nu} \Psi(m_\mu) \otimes \Psi(m_\nu) = 0$$

Because there are no relations between $\Psi(m_\nu)$ for $|\nu| < n$, by the minimality assumption, we can extract the coefficient of $\Psi(m_{(n')})$

$$\sum_{|\mu|=n-n'} c_{\mu \sqcup (n')} \Psi(m_\mu) = 0$$

Extracting the term of $\mu = \lambda^0 \backslash \{n'\}$ gives us $c_{\lambda^0} = 0$ in the above relation. This contradicts our hypothesis, and therefore the map Ψ is injective.

$\square$

6. The Elliptic Hall algebra

6.1. Consider the algebra $\mathcal{E}^+$ generated by elements $u_{k,d}$ for $k \geq 1, d \in \mathbb{Z}$, under the relations:

$$[u_{k_1,d_1}, u_{k_2,d_2}] = 0, \tag{6.1}$$

whenever the points $(k_1, d_1), (k_2, d_2) \in \mathbb{Z}^2$ are collinear, and:

$$[u_{k_1,d_1}, u_{k_2,d_2}] = \alpha_1 \cdot v_{k_1+k_2,d_1+d_2} \tag{6.2}$$

whenever the closed triangle with vertices $(0,0)$, (k_2, d_2) and $(k_1 + k_2, d_1 + d_2)$ is semi-minimal in the sense of Remark 4.5. Here, $\alpha_n \in \mathbb{K}$ are the constants of (4.11), and we define the elements $v_{k,d} \in \mathcal{E}^+$ by the same generating series as in (4.13):

$$\sum_{n=0}^{\infty} x^n v_{na,nb} = \exp\left(\sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha_1} x^n u_{na,nb}\right), \qquad \text{whenever } \gcd(a,b) = 1. \tag{6.3}$$

The algebra $\mathcal{E}^+$ is also bigraded by the two coordinates k and d.

6.2. The existence of minimal triangles we established in Remark 4.5, together with (6.2), imply that the algebra $\mathcal{E}^+$ is generated by $\mathcal{E}^+_1 = \text{span}(u_{1,d}, d \in \mathbb{Z})$. In fact, it was proved in [3] that the map:

$$\Upsilon(u_{1,d}) = P_{1,d} \tag{6.4}$$

induces an isomorphism between $\mathcal{E}^+$ and $\mathcal{A}^+$. We now move to the proof of our main Theorem 1.1, which claims that $\Upsilon(u_{k,d}) = P_{k,d}$ for all $k \geq 1, d \in \mathbb{Z}$.

Proof of Theorem 1.1: Take a minimal triangle with vertices $(0,0), (i, j = s(k,d)_i), (k,d)$ for some $i \in \{1, ..., k-1\}$. By the induction hypothesis and (6.2), we have:

$$Q := \Upsilon(u_{k,d}) = \frac{1}{\alpha_1}[P_{k-i,d-j}, P_{i,j}] \tag{6.5}$$

We need to show that $Q = Q_{k,d}$, and by Proposition 4.14, it is enough to prove the following three claims:

(1) $Q \in \mathcal{B}^{S(k,d)}$

(2) Write $(k,d) = (na, nb)$, where $\gcd(a,b) = 1$. Then for all $x \in \{1, ..., n-1\}$,

$$\Delta_{xa}(Q) = Q_{xa,xb} \otimes Q_{(n-x)a,(n-x)b}$$

(3)

$$\varphi(Q) = \frac{q_1^{-\frac{n}{2}} - q_1^{\frac{n}{2}}}{(1-q^{-1})^k} \cdot \frac{(q_2 - 1)(q_1 - q_2^{-1})}{q_1 - 1}$$

By Lemma 3.6, both $P_{k-i,d-j} * P_{i,j}$ and $P_{i,j} * P_{k-i,d-j}$ have degrees no greater than σ given by:

$$\sigma = s(i,j) + s(k-i, d-j)$$

The sum of degrees is defined by relation (3.10), so there exists some $y \leq i$ and $z \leq k-i$ such that $\sigma_{y+z} = s(i,j)_y + s(k-i,d-j)_z$. Therefore, the lattice point $A = (y+z, \sigma_{y+z})$ lands below the line $(0,0),(k,d)$, **or** inside the paralellogram spanned by the vectors (i,j) and $(k-i,d-j)$:

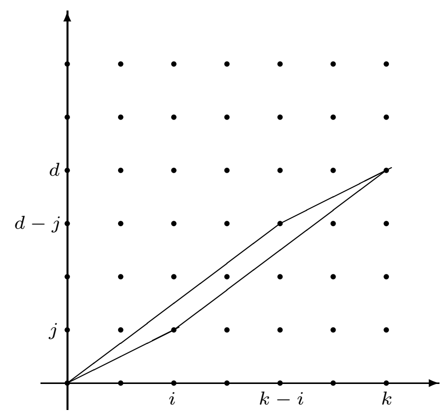


Figure 6.2

By the minimality of our initial triangle, there are no lattice points strictly inside the paralellogram or on the open edges, other than the main diagonal and the vertices. We conclude that:

$$\sigma_{y+z} \leq S(k,d)_{y+z} + \delta^{k-i}_{y+z}, \qquad \forall y \leq i, z \leq k-i \tag{6.6}$$

with the extra δ coming from vertex $(k-i, d-j)$ of the paralellogram, and corresponds to the special case $y = 0$ and $z = k-i$. This special case is the only one where $\Delta_{y+z}([P_{i,j}, P_{k-i,d-j}])$ might have a term of order $O(\xi^{S(k,d)_{k-i}+1})$. However, Lemma 3.6 tells us that the term of this order is precisely $P_{k-i,d-j} \otimes P_{i,j}$ in both $\Delta_{y+z}(P_{k-i,d-j} * P_{i,j})$ and $\Delta_{y+z}(P_{i,j} * P_{k-i,d-j})$, so it drops out when we take the commutator. We conclude that $Q \in \mathcal{B}^{S(k,d)}$, thus proving claim (1).

Claim (2) will use the following Proposition, which will be proved right after we conclude the proof of the Theorem.

Proposition 6.3. *For a semi-minimal triangle with vertices* $(0,0), (i,j = s(k,d)_i), (k,d)$ *such that* $\gcd(k,d) = 1$*, we have:*

$$\Delta_i^{S(k,d)}(P_{k,d}) = Q_{i,j} \otimes Q_{k-i,d-j}$$

With this in mind, we will now compute $\Delta_{xa}(Q)$. We pick up a contribution whenever $xa = y + z$, such that:

$$s(i,j)_y + s(k-i,d-j)_z = S(k,d)_{xa} = xb$$

Looking back at Figure 6.2, we see that this only happens in two cases:

- $(y,z) = (xa-k+i, k-i)$ when $xa > k-i$
- $(y,z) = (0, xa)$ when $xa < k-i$

By Lemma 3.6, the term we pick up in $P_{k-i,d-j} * P_{i,j}$ (respectively, $P_{i,j} * P_{k-i,d-j}$) equals:

$$(P_{k-i,d-j} \otimes 1) * \Delta_{xa-k+i}(P_{i,j}), \qquad \text{(respectively } \Delta_{xa-k+i}(P_{i,j}) * (P_{k-i,d-j} \otimes 1))$$

in the first case, and:

$$\Delta_{xa}(P_{k-i,d-j}) * (1 \otimes P_{i,j}), \qquad \text{(respectively } (1 \otimes P_{i,j}) * \Delta_{xa}(P_{k-i,d-j}))$$

in the second case. So we conclude that:

$$\alpha_1 \Delta(Q) = \sum_{xa > k-i} [P_{k-i,d-j} \otimes 1, \Delta_{i-(n-x)a}(P_{i,j})] + \sum_{xa < k-i} [\Delta_{xa}(P_{k-i,d-j}), 1 \otimes P_{i,j}]$$

By Proposition 6.3, the above become:

$$\Delta(Q) = \frac{1}{\alpha_1} \sum_{xa > k-i} [P_{k-i,d-j}, P_{i-(n-x)a, s(i,j)_{i-(n-x)a}}] \otimes Q_{(n-x)a,(n-x)b} +$$
$$+ \frac{1}{\alpha_1} \sum_{xa < k-i} Q_{xa,xb} \otimes [P_{k-i-xa, s(k-i,d-j)_{k-i-xa}}, P_{i,j}].$$

By applying the induction hypothesis of Theorem 1.1, the above yields:

$$\Delta(Q) = \sum_{xa > k-i} Q_{xa,xb} \otimes Q_{(n-x)a,(n-x)b} + \sum_{xa < k-i} Q_{xa,xb} \otimes Q_{(n-x)a,(n-x)b}.$$

which proves claim (2). To prove claim (3), the multiplicativity property of Proposition 2.8 implies that:

$$\varphi(Q) = \frac{1}{\alpha_1} \varphi(P_{i,j}) \varphi(P_{k-i,d-j}) \left[q^{(id-jk)/2} - q^{(jk-id)/2} \right]$$

By Pick's Theorem, we have $id - jk = n$. Formula (4.10) and the minimality of the triangle implies that

$$\varphi(P_{i,j}) = \alpha_1 \cdot \frac{(1-q^{-1})^{-i}}{q_1^{1/2} - q_1^{-1/2}}, \qquad \varphi(P_{k-i,d-j}) = \alpha_1 \cdot \frac{(1-q^{-1})^{i-k}}{q_1^{1/2} - q_1^{-1/2}}$$

and therefore:

$$\varphi(Q) = \frac{1}{\alpha_1} \cdot \frac{\alpha_1^2}{(1-q^{-1})^k (q_1^{1/2} - q_1^{-1/2})^2} (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) = \frac{(q^{-n/2} - q^{n/2})(q_2 - 1)(q_1 - q_2^{-1})}{(1-q^{-1})^k (q_1 - 1)}$$

This proves claim (3), and with it, the Theorem.

□

Proof of Proposition 6.3: We may assume that $\gcd(i,j) = n$ for $n \geq 1$ and $\gcd(k-i, d-j) = 1$ (the opposite case is analogous). Let us write $(i,j) = (na, nb)$. Then by (4.12), we have:

$$P_{k,d} = (1-q^{-1})^{1-k} \mathrm{Sym} \left[\frac{z_1 \prod_{j=1}^k z_j^{S_j - S_{j-1}}}{z_k \left(\frac{z_1}{z_2} - q\right) \dots \left(\frac{z_{k-1}}{z_k} - q\right)} \prod_{i<j} \omega\left(\frac{z_j}{z_i}\right) \right]$$

where $S_x = S(k,d)_x = \left\lfloor \frac{xd}{k} \right\rfloor$. To compute $\Delta_i^S(P_{k,d})$, we need to perform an analysis similar to the one in the proof of Proposition 5.4, where we multiply $z_1, ..., z_i$ by ξ and compute the term of order ξ^{S_i}. [5] Just like in the proof of the proposition, we get a relevant term whenever:

$$x_1 < y_1 < x_2 < y_2 < \dots < x_t < y_t \in \{0, ..., k\} \tag{6.7}$$

are such that:

$$\sum_{j=1}^{y} (y_j - x_j) = i, \qquad \sum_{j=1}^{t} \left(S_{y_j} - S_{x_j}\right) - t + \delta_{x_1}^0 = S_i$$

Because there are no lattice points inside the parallelogram (6.2) (neither on the main diagonal, since $\gcd(k,d) = 1$), the following inequality holds:

$$S_y - S_x - 1 + \delta_x^0 = \left\lfloor \frac{yd}{k} \right\rfloor - \left\lfloor \frac{xd}{k} \right\rfloor - 1 + \delta_x^0 \leq \frac{y-x}{i} \left\lfloor \frac{id}{k} \right\rfloor = \frac{y-x}{i} \cdot S_i$$

and it only becomes an equality when $(x,y) = (0, za)$ or $(k-i+za, k)$ for some $z \in \{0, ..., n\}$. Therefore, the only configurations (6.7) which contribute to the leading term are $t = 1$, $x_1 = 0$, $y_1 = na$, **and** $t = 1$, $x_1 = k - na$, $y_1 = k$ **and** $t = 2$, $x_1 = 0$, $y_1 = za$, $x_2 = k - i + za$, $y_2 = k$. The corresponding terms are equal to:

$$(1-q^{-1})^{1-k} \left(X_{i,j}^{(0^{n-1})} - q^{-1} X_{i,j}^{(1^{n-1})} - q^{-1} \sum_{y+z=n-2}^{y,z \geq 0} X_{i,j}^{(0^z)} * X_{i,j}^{(1^w)} \right) \otimes X_{k-i,d-j}$$

By (5.9), the above equals:

$$(1-q^{-1})^{2-k} \sum_{z+w=n-2}^{z,w \geq 0} X^{(0^z 1^w)} \otimes X_{k-i,d-j}$$

By (5.11), the above is precisely equal to $Q_{i,j} \otimes Q_{k-i,d-j}$, as desired.

[5] The difference between this case and the one in Proposition 5.4 is that i is no longer a multiple of $\frac{k}{\gcd(k,d)}$

□

7. Doubles

7.1. Now consider an infinite set of variables $z_{-1}, z_{-2}, ...$, and define the shuffle algebra:

$$\mathcal{A}^- \subset \bigoplus_{k\geq 0} \mathrm{Sym}_{\mathbb{K}}(z_{-1}, ..., z_{-k}),$$

consisting of rational functions which satisfy (2.5) and (2.6). We will slightly change the product to:

$$P(z_{-1}, ..., z_{-k}) * Q(z_{-1}, ..., z_{-l}) =$$

$$= \mathrm{Sym}\left[P(z_{-1}, ..., z_{-k})Q(z_{-(k+1)}, ..., z_{-(k+l)})\prod_{i=1}^{k}\prod_{k+1=j}^{k+l}\omega\left(\frac{z_{-j}}{z_{-i}}\right)\right]$$

although everything we have discussed so far goes through with no modifications. In fact, sending $P(z_1, ..., z_k)$ to $P(z_{-1}^{-1}, ..., z_{-k}^{-1})$ gives an isomorphism $\mathcal{A}^+ \cong \mathcal{A}^-$. By analogy with section 4.9, we can define **minimal shuffle elements**:

$$P_{-k,d} = (1-q^{-1})^{1-k}\frac{(q_1-1)(q_2-1)}{(q_1^n-1)(q_2^n-1)}$$

$$\sum_{x=0}^{n-1} q^{n-x-1}\cdot \mathrm{Sym}\left[\frac{\prod_{j=1}^k z_j^{\left\lfloor \frac{jd}{k}\right\rfloor - \left\lfloor \frac{(j-1)d}{k}\right\rfloor}\cdot z_{a+1}...z_{xa+1}}{\left(\frac{z_2}{z_1}-q\right)...\left(\frac{z_k}{z_{k-1}}-q\right)\cdot z_a...z_{xa}}\prod_{i<j}\omega\left(\frac{z_j}{z_i}\right)\right]$$

for all $(k,d) = (na, nb)$ with $\gcd(a,b) = 1$.

7.2. Recall that Λ denotes the $\mathbb{K}-$algebra of symmetric polynomials in infinitely many variables, and we will consider:

$$\mathcal{A}^0 = \mathrm{Sym}^2_{\mathbb{K}}(\Lambda)$$

It consists of two commuting copies of Λ, the generators of which will be denoted by p_n (respectively h_n) and p_{-n} (respectively h_{-n}). It will sometimes be handy to work with the generating series:

$$h^{\pm}(z) = \sum_{n\geq 0} h_{\pm n} z^{\mp n} = \exp\left(\sum_{n\geq 1} p_{\pm n} z^{\mp n}\right) = \exp\left(p^{\pm}(z)\right) \tag{7.1}$$

7.3. Then define the **double shuffle algebra**:

$$\mathcal{A} = \mathcal{A}^- \otimes \mathcal{A}^0 \otimes \mathcal{A}^+$$

on which we impose the relations:

$$[p_{\pm n}, P(z_{\pm' 1}, ..., z_{\pm' k})] = \mp \pm' \alpha_n \cdot P(z_{\pm' 1}, ..., z_{\pm' k})(z_{\pm' 1}^{\pm n} + ... + z_{\pm k}^{\pm' n}) \in \mathcal{A}^{\pm} \tag{7.2}$$

for all $P \in \mathcal{A}^{\pm}$, and:

$$[P(z_1, ..., z_k), Q(z_{-1}, ..., z_{-l})] = \sum_{a=1}^{\min(k,l)} \alpha_1^a : \mathrm{Res} : \left[\prod_{1 \leq i \neq j \leq a} \omega^{-1}\left(\frac{w_i}{w_j}\right) \right.$$

$$\left. \frac{h(w_1)}{w_1} ... \frac{h(w_a)}{w_a} \cdot \frac{Q(z_{-1}, ..., z_{-(l-a)}, w_1, ..., w_a)}{\prod_{i=1}^a \prod_{j=1}^{l-a} \omega\left(\frac{z_{-j}}{w_i}\right)} * \frac{P(z_1, ..., z_{k-a}, w_1, ..., w_a)}{\prod_{i=1}^a \prod_{j=1}^{k-a} \omega\left(\frac{w_i}{z_j}\right)} \right] \tag{7.3}$$

for any $P \in \mathcal{A}^+$ and $Q \in \mathcal{A}^-$, where Res denotes the residue at 0 minus the residue at ∞, in the order $w_1, ..., w_a$. Depending on whether we are taking the residue at 0 or ∞, we need to put the sign $-$ or $+$ above each $h(w_i)$ to give us the correct expansion.

7.4. To complete the picture, we need to explain what is meant by the "normal ordered residue" : Res : of relation (7.3). This is computed as follows. As was mentioned in subsection 5.1, the shuffle element Q can be written as a linear combination of:

$$\mathrm{Sym}\left[z_{-1}^{\mu_1} ... z_{-l}^{\mu_l} \prod_{1 \leq i < j \leq l} \omega\left(\frac{z_{-j}}{z_{-i}}\right) \right] \tag{7.4}$$

for $\mu_1, ..., \mu_l \in \mathbb{Z}$. In formula (7.3), we need to set some of these variables equal to $w_1, ..., w_a$. By definition, the meaning of the normal ordered residue : Res : in (7.3) is that we only take the summands of (7.4) which keep the variables $w_1, ..., w_a$ in order. The shuffle element P is not affected by this convention.

7.5. The main algebraic object of [3] is the algebra $\mathcal{E}$, which has generators $u_{k,d}$ for all $(k, d) \in \mathbb{Z}^2 \backslash (0, 0)$, under the following slight generalizations of (6.1) and (6.2):

$$[u_{k_1,d_1}, u_{k_2,d_2}] = \delta_{k_2}^{-k_1} \delta_{d_2}^{-d_1} (h_{+0}^{k_1} - h_{-0}^{k_1}) \frac{\alpha_1^2}{\alpha_{\gcd(k_1,d_1)}} \tag{7.5}$$

whenever the points $u_{k_1,d_1}, u_{k_2,d_2} \in \mathbb{Z}^2$ are collinear, and:

$$[u_{k_1,d_1}, u_{k_2,d_2}] = \alpha_1 \cdot v_{k_1+k_2,d_1+d_2} \tag{7.6}$$

whenever the closed, oriented triangle with vertices $(0, 0), (k_2, d_2), (k_1 + k_2, d_1 + d_2)$ contains no lattice points except its vertices and the edge $(0, 0), (k_1 + k_2, d_1 + d_2)$.

Here, the elements $v_{k,d}$ are given by (6.3). This new algebra is a double of $\mathcal{E}^+$.

7.6. Our main result concerning the doubles $\mathcal{E}$ and $\mathcal{A}$ is the following generalization of Theorem 1.1:

Theorem 7.7. *The assignment:*

$$\Upsilon(u_{-k,d}) = P_{-k,d}, \qquad \Upsilon(u_{k,d}) = P_{k,d}, \qquad \forall k \geq 1, d \in \mathbb{Z}$$

$$\Upsilon(u_{0,d}) = -\frac{\alpha_1}{\alpha_d} p_d, \qquad \Upsilon(u_{0,-d}) = -\frac{\alpha_1}{\alpha_d} p_{-d}, \qquad \forall d \geq 1$$

gives an isomorphism $\Upsilon : \mathcal{E} \cong \mathcal{A}$.

Proof In Proposition 1.1 of [3], it is showed that:

$$\mathcal{E} = \mathcal{E}^- \otimes \mathcal{E}^0 \otimes \mathcal{E}^+$$

where:

$$\mathcal{E}^{\pm} = \langle u_{\pm k,d}, k \geq 1, d \in \mathbb{Z} \rangle, \qquad \mathcal{E}^0 = \langle u_{0,d}, d \neq 0 \rangle,$$

under the following subset of the relations (7.5) and (7.6):

$$[u_{0,\pm d}, u_{\pm' 1,d'}] = \pm \pm' \alpha_1 \cdot u_{\pm' 1, d' \pm d}$$

and

$$[u_{1,d}, u_{-1,d'}] = \alpha_1 \left(\delta_{d+d' \geq 0} \cdot v_{0,d+d'} - \delta_{d+d' \leq 0} \cdot v_{0,d+d'} \right)$$

These relations are simply (7.2) and (7.3) [6] for $k = l = 1$, and therefore Υ gives a well-defined algebra homomorphism $\mathcal{E} \longrightarrow \mathcal{A}$. Relation (7.5) implies that $\mathcal{E}^0 \cong \mathrm{Sym}^2_{\mathbb{K}}(\Lambda) \cong \mathcal{A}^0$, while a trivial modification of Theorem 1.1 tells us that Υ restricted to $\mathcal{E}^{\pm}$ maps it isomorphically onto $\mathcal{A}^{\pm}$. This implies the injectivity of the map Υ, while the surjectivity is equivalent to Conjecture 2.6.

□

[6]When $d + d' = 0$ in the above, the right hand side becomes $\gamma(h_{-0} - h_{+0})$; note that these two generators are different by (7.1)

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